P. I. Fomin Memorial Seminar

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Dark Matter in $f(R)$ Gravity

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$f(R)$ gravity theory

No new fields but one extra scalar degree of freedom

Series expansion of the action:

units $\hbar = c = 1$, signature $(-,+,+,+)$

$$L_g = \frac{M^2}{3}\left(-2\Lambda + R + \frac{R^2}{6m^2} + \dots\right) = \frac{M^2}{3} f(R)$$

Planck mass $M = \sqrt{3/16\pi G} \approx 3 \times 10^{18}\,\text{GeV}$

Cosmological constant $\Lambda \approx \left(3 \times 10^{-33}\,\text{eV}\right)^2$

- For $m \simeq 10^{-5} M \simeq 3 \times 10^{13}$ GeV, we are dealing with the Starobinsky inflationary model
- We assume *much smaller* values of m to describe *dark matter* instead: meV $\lesssim m \lesssim$ MeV

Legendre transform and the scalaron

Jordan frame → Einstein frame

Original action

$$S_{\rm g} = \frac{M^2}{3}\int d^4x\sqrt{-g}\, f(R)$$
$$= \frac{M^2}{3}\int d^4x\sqrt{-g}\,[\Omega R - \mu(\Omega)]$$

Assuming $f'(R) > 0, \; f''(R) > 0$

$$f'(R) = \Omega \quad \Rightarrow \quad R = R(\Omega)$$
$$\mu(\Omega) = [\Omega R - f(R)]_{R=R(\Omega)}$$

Conformal transformation:

$$g_{\mu\nu} = \Omega^{-1}\tilde{g}_{\mu\nu}\,, \quad \Omega = e^{\phi/M}$$

Transformed action

$$S_{\rm g} = \frac{M^2}{3}\int d^4x\sqrt{-\tilde{g}}\,\tilde{R}$$
$$-\int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}\left(\tilde{\nabla}\phi\right)^2 + V(\phi)\right]$$

$$V(\phi) = \frac{M^2}{3}e^{-2\phi/M}\mu\left(e^{\phi/M}\right)$$

This is the so-called *Einstein frame* of field variables

Extremal points of the potential $V(\phi)$ correspond to R satisfying

$$R\,f'(R) = 2f(R)$$

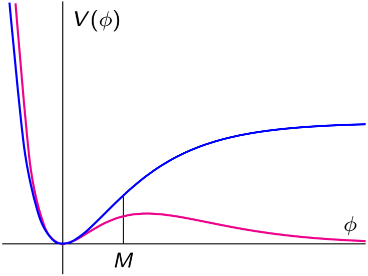
Gravitational action in the Einstein frame

Conformal transformation $g_{\mu\nu} = e^{-\phi/M}\widetilde{g}_{\mu\nu}$

$$S_g = \frac{M^2}{3}\int d^4x\sqrt{-\widetilde{g}}\,\widetilde{R} - \int d^4x\sqrt{-\widetilde{g}}\left[\frac{1}{2}\widetilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]$$

$$f(R) = -2\Lambda + R + \frac{R^2}{6m^2}\,, \qquad V(\phi) = \frac{1}{2}m^2M^2\left(1 - e^{-\phi/M}\right)^2 + V_\Lambda(\phi)$$

$$f(R) = -2\Lambda + \frac{R}{1 - R^2/6m^2}\,, \quad V(\phi) = 2M^2m^2e^{-\phi/M}\left(1 - e^{-\phi/2M}\right)^2 + V_\Lambda(\phi)$$



$$V_\Lambda(\phi) = \frac{2}{3}\Lambda M^2 e^{-2\phi/M}$$

Neglect the cosmological constant Λ:

$$V(\phi) \approx \frac{1}{2}m^2\phi^2$$

around the minimum

Lower bound on the mass m

(from now on we work in the Einstein frame and remove tildes)

In matter Lagrangian, $L_{\text{m}}\left(e^{-\phi/M} g_{\mu\nu}, \Psi\right)$:

$$\left(\Box + m^2\right)\phi = -\frac{1}{2M} T^{\mu}{}_{\mu}$$

Exchange of the scalaron ϕ leads to the presence of additional Yukawa forces (Stelle, 1978)

$$\Phi_{\text{grav}} = -\frac{2G\mathcal{M}}{r}\left(1 + \frac{1}{3}e^{-mr}\right), \qquad \frac{1}{m} \approx \left(\frac{2 \times 10^{-4}\,\text{eV}}{m}\right)\text{mm}$$

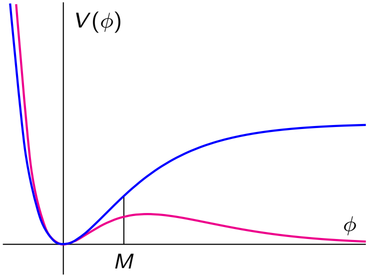
Current estimates give the lower bound on m (Kapner *et al.*, 2007; Adelberger *et al.*, 2007; Perivolaropoulos and Kazantzidis, 2019)

$$m \geq 2.7 \times 10^{-3}\,\text{eV} \quad \text{at 95\% C.L.}$$

Scalaron as a dark-matter candidate
(Cembranos, 2008)

Einstein frame:

$$S_g = \frac{M^2}{3} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right]$$



The scalaron ϕ, in principle, can play the role of dark matter (Cembranos, 2008). Consideration of its decay gives an upper bound

$$\boxed{m \lesssim 1\,\text{MeV}} \qquad \text{(discussed below)}$$

How are the initial conditions formed?

The Higgs sector of the Standard Model

The only sector not invariant with respect to conformal transformations

Yu. S., Physics Letters B **820**, 136469 (2021)

$$S_{\mathrm{H}} = -\int d^4x\sqrt{-g}\left[g^{\mu\nu}\left(D_\mu\Phi\right)^\dagger D_\nu\Phi + \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2\right], \quad \Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$v \approx 246\ \mathrm{GeV}, \quad m_{\mathrm{H}} = \sqrt{2\lambda}v \approx 125\ \mathrm{GeV}, \quad \lambda \approx 0.13$$

Conformal transformation (Jordan frame $\to$ Einstein frame):

$$g_{\mu\nu} \to e^{-\phi/M} g_{\mu\nu} \qquad \Phi \to e^{\phi/2M}\Phi \qquad \psi \to e^{3\phi/4M}\psi$$

$$S_{\mathrm{H}} = -\int d^4x\sqrt{-g}\left[\left(D_\mu\Phi\right)^\dagger D^\mu\Phi + \frac{1}{2M}\partial_\mu\left(\Phi^\dagger\Phi\right)\partial^\mu\phi + \frac{1}{4M^2}\Phi^\dagger\Phi\,\partial_\mu\phi\partial^\mu\phi\right.$$

$$\left. + \lambda\left(\Phi^\dagger\Phi - e^{-\phi/M}\frac{v^2}{2}\right)^2\right]$$

Interaction is suppressed by powers of M^{-1}

Mixing between the Higgs field and the scalaron

- Unitary gauge: $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ h\end{pmatrix}$, $\quad h = v + \varphi$

  Quadratic part of the Lagrangian $L_\phi + L_{\rm H}$:

$$L_2 = -\frac{1}{2}\left(\partial\varphi\right)^2 - \frac{1}{2}\left(1+\frac{v^2}{4M^2}\right)\left(\partial\phi\right)^2 - \frac{v}{2M}\partial\varphi\partial\phi - \lambda v^2\left(\varphi + \frac{v}{2M}\phi\right)^2 - \frac{1}{2}m^2\phi^2$$

- A shift $\boxed{\varphi = \chi - \frac{v}{2M}\phi}$ Note that $\frac{v}{2M} \sim 10^{-16}$

$$L_2 = -\frac{1}{2}\left(\partial\chi\right)^2 - \frac{1}{2}\left(\partial\phi\right)^2 - \lambda v^2\chi^2 - \frac{1}{2}m^2\phi^2\,, \qquad m_\chi = \sqrt{2\lambda}v$$

Coupling of the scalaron to other fields:

$$-\gamma h\bar{\psi}\psi \;\to\; \frac{\gamma v}{2M}\phi\bar{\psi}\psi = \frac{m_\psi}{2M}\phi\bar{\psi}\psi\,, \quad -\frac{m_W^2}{M}\phi W^+_\mu W^{-\mu} - \frac{m_Z^2}{2M}\phi Z_\mu Z^\mu$$

Anomalous couplings and the scalaron decays

Conformal transformation of matter variables $q \to q'$ in the path integral does not leave the integration measure invariant (Fujikawa's (1979) method of obtaining quantum anomalies):

$$\int e^{iS[q]}\, Dq = \int e^{iS'[q']}\, J[q']\, Dq' = \int e^{i\left(S'[q'] + \Delta S'[q']\right)}\, Dq'$$

Anomalous couplings between the scalaron and the gauge fields:

$$\Delta \mathcal{L}' = \left(C \alpha_{\rm g} \frac{\phi}{M} + \dots \right) \operatorname{tr} F^2 \,, \quad \alpha_{\rm g} = \frac{g_{\rm g}^2}{4\pi} \quad \text{is the gauge coupling constant}$$

They allow for the scalaron decay into photons, with lifetime

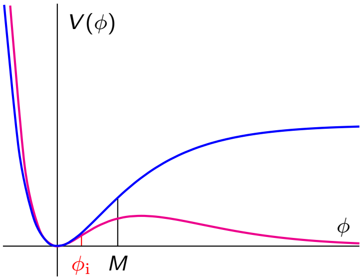
$$\tau \sim \frac{M^2}{\alpha_{\rm em}^2 m^3} \sim 10^{36} \left(\frac{\rm eV}{m} \right)^3 \text{yr}$$

Compare this with the age of the universe 1.4×10^{10} yr. Non-observation of the possible γ flux from the scalaron dark-matter decays gives the upper bound

$$\boxed{m \lesssim 1\,\text{MeV}} \qquad \text{(Cembranos, 2008)}$$

The issue of initial conditions

very similar to the case of axion dark matter



$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

Cembranos, 2008: The scalaron field is "frozen" at some value $\phi_{\rm i}$ as $3H \gg m$ and starts oscillating as $3H \lesssim m$. There are two free parameters in this scenario, m and $\phi_{\rm i}$, which can be tuned so as to obtain the presently observed dark-matter density.

In order to obtain predominantly *adiabatic* dark-matter density perturbations ($\rho_{\rm dm}/s = \text{const}$, supported by observations of the CMB anisotropy), one must ensure that $\phi_{\rm i}$ have small variation over the observable part of the universe at high temperatures:

$$\left\langle \left(\frac{\delta\rho_\phi}{\rho_\phi} \right)^2 \right\rangle \lesssim 0.02 \left\langle \left(\frac{\delta\rho}{\rho} \right)^2 \right\rangle \sim 10^{-11}$$

Inflation and initial conditions for the scalaron

Y.S., arXiv:2207.00267 (July, 1)

Inflaton ζ in the Jordan frame

$$L_{\rm infl} = -\frac{1}{2}(\partial\zeta)^2 - W(\zeta)\,,$$

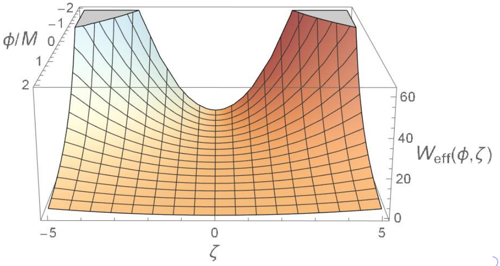
In the Einstein frame, after the conformal transformation $\zeta \to \Omega^{1/2}\zeta$, it takes the form:

$$L_{\rm infl} = -\frac{1}{2}(\partial\zeta)^2 - \frac{1}{2M}\zeta\,(\partial\zeta\partial\phi) - \frac{1}{8M^2}\zeta^2\,(\partial\phi)^2 - \underbrace{e^{-2\phi/M}W\left(e^{\phi/2M}\zeta\right)}_{W_{\rm E}(\phi,\zeta)}$$

Example: $W(\zeta) = \frac{1}{2}m_\zeta^2\zeta^2$:

$$W_{\rm E}(\phi,\zeta) = \frac{1}{2}m_\zeta^2 e^{-\phi/M}\zeta^2$$

$$\langle\delta\phi^2\rangle \simeq \left(\frac{H_{\rm inf}}{2\pi}\right)^2$$



Post-inflationary evolution of the scalaron

$$L_{\rm eff} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial h)^2 - \underbrace{\frac{1}{2M}h\,(\partial h\partial\phi) - \frac{1}{8M^2}h^2(\partial\phi)^2}_{\text{negligible interaction}}$$

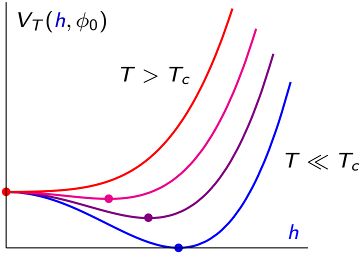
$$\underbrace{-\,V(\phi) - \frac{\lambda}{4}\left(h^2 - e^{-\phi/M}v^2\right)^2}_{-\text{ joint potential } V(\phi,h)} \underbrace{-\,\frac{1}{6}T^2h^2 + \frac{1}{100}Th^3}_{\text{thermal correction}}$$

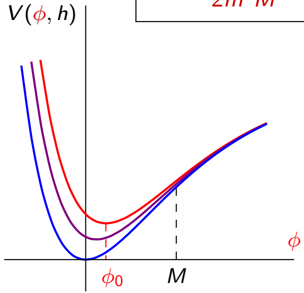
Small parameter:

$$\frac{v^2}{M^2} \sim 10^{-32}$$

$$T_c \approx \sqrt{3\lambda}\,v \approx 154 \text{ GeV}$$

$$\phi_0 \approx \frac{\lambda v^4}{2m^2 M} \ll M$$





Adiabatic invariant

$$\ddot{\phi} + 3H\dot{\phi} + m^2\left[\phi - \phi_0(h)\right] = 0\,, \qquad \phi_0(h) = \phi_0\left(1 - \frac{h^2}{v^2}\right) = \phi_0\frac{T^2}{T_c^2}$$

The scalaron remains frozen until $3H \simeq m$, after which it starts oscillating

Adiabatic invariant: $$I = \frac{a^3}{2}\left[\dot{\xi}^2 + m^2\xi^2\right] \approx \text{const}$$

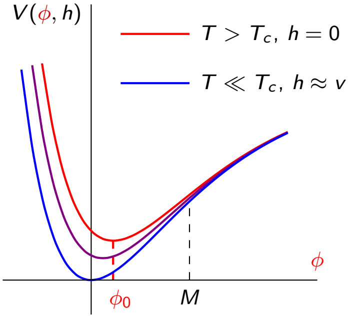
$$\boxed{\xi \equiv \phi - \phi_0(h)}$$

The jump at the beginning of the electroweak crossover:

$$\dot{\phi}_0\Big|_{T=T_c} = -2H_c\phi_0$$

Two limiting cases for the amplitude of ξ:

$$\xi_i \ll \phi_0 \quad \text{and} \quad \xi_i \gg \phi_0$$



Scenario with special initial conditions

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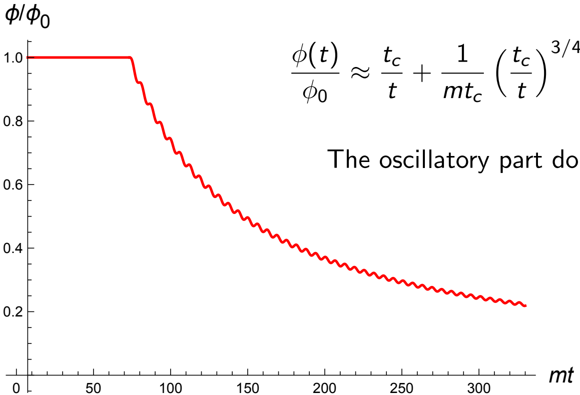
The scalaron is excited at the beginning of the electroweak crossover

After the electroweak crossover: $I = 2a_c^3 H_c^2 \phi_0^2$



$$\frac{\phi(t)}{\phi_0} \approx \frac{t_c}{t} + \frac{1}{mt_c}\left(\frac{t_c}{t}\right)^{3/4} \sin\left[m\left(t - t_c\right)\right]$$

The oscillatory part dominates at $T \lesssim 30\,\mathrm{MeV}$

Dark matter abundance

Energy density in the scalaron oscillations:

$$\rho_\phi = I/a^3 = 2H_c^2\phi_0^2\left(\frac{a_c}{a}\right)^3$$

$$\phi_0 \approx \frac{\lambda v^4}{2m^2 M} = 4.8\times 10^{12}\ \mathrm{GeV}$$

The adiabaticity constraint:

$$\left\langle\left(\frac{\delta\rho_\phi}{\rho_\phi}\right)^2\right\rangle \lesssim 10^{-11}$$

$$\Downarrow$$

$$H_{\rm inf} \lesssim 10^7\ \mathrm{GeV}\,, \qquad \boxed{T_{\rm r} \lesssim 10^{12}\ \mathrm{GeV}}$$

$$H_c^2 = \frac{\pi^2 g_c T_c^4}{60M^2}\,, \qquad \left(\frac{a_c}{a_0}\right)^3 = \frac{2T_0^3}{g_c T_c^3}\,,$$

where g_c is the number of relativistic degrees of freedom at thermal equilibrium (this cancels anyway)

$$\Omega_\phi H_0^2 = \frac{\rho_0}{2M^2} = \frac{\pi^2\lambda^{5/2}v^9 T_0^3}{10\sqrt{3}\,M^6 m^4}$$

$$\boxed{\Omega_\phi h_{100}^2 \simeq 0.12\left(\frac{4.4\times 10^{-3}\,\mathrm{eV}}{m}\right)^4}$$

$$h_{100} = \frac{H_0}{100\ \mathrm{km/s\,Mpc}}$$

Scenario with generic initial conditions

The field starts oscillating as $3H \simeq m$ (moment 'i')

$$\rho_0 = \rho_i \left(\frac{a_i}{a_0}\right)^3 = \frac{m^2 \xi_i^2}{2} \cdot \frac{2\, T_0^3}{g_i\, T_i^3} = \left(\frac{3\pi^2}{5}\right)^{3/4} \frac{m^{1/2} \xi_i^2\, T_0^3}{g_i^{1/4} M^{3/2}}$$

$$\boxed{\Omega_\phi h_{100}^2 = 0.12 \left(\frac{100}{g_i}\right)^{1/4} \left(\frac{m}{\text{eV}}\right)^{1/2} \left(\frac{1.3 \times 10^7\, \xi_i}{M}\right)^2}$$

Lower bound on m leads to an upper bound

$$\frac{\xi_i}{M} \lesssim 3 \times 10^{-7}$$

The adiabaticity constraint:

$$\left\langle \left(\frac{\delta\rho_\phi}{\rho_\phi}\right)^2 \right\rangle = \frac{4 \left\langle \delta\xi_i^2 \right\rangle}{\xi_i^2} = \left(\frac{H_{\text{inf}}}{\pi \xi_i}\right)^2 \lesssim 10^{-11}$$

$$\boxed{T_r \lesssim 10^{12} \left(\frac{\text{eV}}{m}\right)^{1/8} \text{GeV}}$$

Yukawa gravitational forces

The smallness of the direct specific gravitational manifestations would make it difficult to establish that we are dealing with $f(R)$ gravity. Perhaps, this could be done by detecting a specific Yukawa contribution to gravitational forces at submillimetre spatial scales (Stelle, 1978)

$$\Phi_{\rm grav} = -\frac{2GM}{r}\left(1 + \frac{1}{3}e^{-mr}\right), \qquad \frac{1}{m} \approx \left(\frac{2\times 10^{-4}\,{\rm eV}}{m}\right)\,{\rm mm}$$

Current estimates give (Kapner *et al.*, 2007; Adelberger *et al.*, 2007; Perivolaropoulos and Kazantzidis, 2019)

$$m \geq 2.7\times 10^{-3}\,{\rm eV} \quad \text{at 95\% C.L.}$$

This is comparable to our lower bound $\boxed{m \gtrsim 4\times 10^{-3}\,{\rm eV}}$

Conclusion

- The scalaron in $f(R)$ gravity can be a cold dark matter candidate similar to an axion if its mass is in the range $4\ \mathrm{meV} \lesssim m \lesssim 1\ \mathrm{MeV}$.
- The scalaron is always in the regime $|\phi| \ll M \ \Rightarrow$ quantum corrections to the scalaron potential are small, and the scalaron with this mass is practically "sterile"
- Since the scalaron was used to describe dark matter, inflation must be ensured by some other field (this involves a very high-energy physics)
- Inflationary constraints on this scenario are rather weak; essentially, $T_{\mathrm{r}} \lesssim 10^{12}\ \mathrm{GeV}$
- The theory could be probed by detecting Yukawa gravitational force at sub-millimetre spatial distances

THANK YOU